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AN INTERESTING MATERIAL THAT APPEARS TO BE FIT TO POSSIBLY ALL FUTURE MECHANICAL VIBRATION TEXTBOOKS

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Abstract. A material is suggested for future mechanical vibration textbooks. Both mathematically and conceptually it is simpler than most of the material that is already included in the existing textbooks. It pertains to the inverse vibration problem for inhomogeneous beam, i.e. the beam with the modulus of elasticity that varies along the axial coordinate. Specifically, the solution of the following problem is presented: Find a distribution of the modulus of elasticity of an inhomogeneous beam such that the beam would possess the preselected simple, polynomial vibration mode shape.

1. INTRODUCTION

The equation of the vibration of the uniform and homogeneous beam

$$EI\frac{\partial^4 w}{\partial x^4} + \rho\frac{\partial^2 w}{\partial t^2} = 0 \tag{1}$$

was first derived by Jacob Bernoulli and Leonhard Euler in 1730s. In Eq. (1) $w(x,t)$ is the transverse displacement, $x=$ axial coordinate, $t=$ time, E = modulus of elasticity, I = moment of inertia, ρ = mass density, A = cross-sectional area.

Since then the solution of Eq. (1) for uniform beams for various boundary conditions became a classic, and is rightfully included perhaps in all structural vibration textbooks (see e.g., Timoshenko *et al.* 1974; Rao, 1995, Meirovitch, 2001).

We will recapitulate that solution briefly to define the motivation of this study. First of all we look for harmonic vibrations in time, getting

$$w(x,t) = Y(x)e^{i\omega t}, \tag{2}$$

where $Y(x)$ is the vibration mode, ω = natural frequency. Both are saught in conjunction with the solution of Eq. (1) satisfying the appropriate boundary conditions. Substitution of Eq. (2) into Eq. (1) leads to the ordinary differential equation for the vibration mode $Y(x)$:

$$EI\frac{d^4 Y}{dx^4} - \rho A\omega^2 Y = 0 \tag{3}$$

We introduce a parameter

$$k^4 = \frac{\rho A \omega^2}{EI} \tag{4}$$

so that Eq. (2) becomes

$$Y^{IV} - k^4 Y = 0. \tag{5}$$

All too familiar solution form

$$Y(x) = C \exp(rx) \tag{6}$$

leads to the characteristic equation

$$r^4 - k^4 = 0 \tag{7}$$

with roots

$$r_{1,2} = \pm ik, \qquad r_{3,4} = \pm k. \tag{8}$$

The solution of Eq. (6) reads

$$Y(x) = C_1 e^{ikx} + C_2 e^{-ikx} + C_3 e^{kx} + C_4 e^{-kx} \tag{9}$$

or, in terms of trigonometric and hyperbolic functions

$$Y(x) = D_1 \sin kx + D_2 \cos kx + D_3 \sinh kx + D_4 \cosh kx. \tag{10}$$

To find k, and hence the sought natural frequency ω, as well as the mode shape $Y(x)$ we ought to satisfy the boundary conditions. This leads to the transcendental equations that are listed below:

$$\begin{array}{ll} P-P: & \sin(kL) = 0, \quad k = \pi/L \\ G-P: & \cosh(kL) = 0 \\ C-F: & \cosh(kL) = -1 \\ C-G: & \tan(kL) + \tanh(kL) = 0 \\ C-P: & \tan(kL) = \tanh kL \\ C-C: & \cos(kL)\cosh(kL) = 1 \end{array} \tag{11}$$

where "P" stands for the pinned end, "C" signifies the clamped end, "F" denotes free end, whereas "G'" is associated with the guided end; L denotes the length of the beam. For the uniform beam that either simply supported or sliding at both ends the closed form solution is available for the fundamental natural frequency, reading

$$\omega_1^2 = \pi^4 EI/\rho A L^4. \tag{12}$$

Now we pose the following question: Are these three cases, i.e. beams with any combination of a simply supported and or a sliding end, the only ones that lead to the closed-form solutions?

A German proverb maintains: "Don't ask questions for you may well get an answer". Scientists naturally are not afraid to pose questions for they are looking for answers, be they positive or negative. Likewise, we cannot be guided by the following American proverb: "Ask no questions and get no lies", for we do anticipate to get correct replies to our queries. Likewise, and English proverb "Don't ask questions about fairy tales" is inapplicable for in our case the search for simple solutions is a real quest. If the reply to our inquiry is negative we will learn that no simpler solution exist and that Euler's solution is the simplest one. If however, our search will lead to the affirmative reply, we may be rewarded by deriving a novel solution that may find its place in hopefully all or at least some future texts.

Therefore, after the brief review of three above negative proverbs, we embrace positive ones: "Questioning is the door of knowledge" (Irish proverb) and "He that nothing questions, nothing learns" (English proverb). Hereinafter, we submit a problem that is conceptually different from Euler's above problem, and leads to a simple, and elegant solution.

An additional angle of looking at things can be derived once we note that the above problem is a direct boundary-value problem. In other words, we have assumed that the material and geometric properties are known and we are looking for the spectral characteristics: the mode shape $\Upsilon(x)$ and the natural frequency ω. An alternative formulation involves the knowledge of some of the output characteristics and looking for the cause. In particular, we pose the following question: Is there a beam that possesses a preselected fundamental mode shape? This question, in perfect analogy to many inverse problems, may have no answer; it may possess an unique answer, or an infinite amount of affirmative replies. We hereinafter will follow "Okham's razor", maintaining that of a problem can be explained by simple means, than the use of more complex explanation is done in vain. We will consider simplest class of functions, namely the polynomials, and specify our question: "Is there a beam that possesses a preselected polynomial mode shape?" We already have seen that the *uniform* beams have mode shape that are combinations of trigonometric and hyperbolic functions. Thus, the inspection of the direct problem's solutions provides a negative answer to our inquiry. Therefore a class of problems in which we will look for the polynomial mode shapes ought to be enlarged. It is suggested herein that we ought to look at an inhomogeneous and/or a nonuniform beam. Inhomogeneity involves a variation of either modulus of elasticity and/or material density. In other words, we will be concerned with the cause, namely, with either E or ρ or both depending upon the axial coordinate x. Nonuniformity involves a variation of the cross sectional area A and/or of the moment of inertia I. For specifity we will limit ourselves with inhomogeneous but uniform case, i.e. when E and/or ρ vary along beam's axis while the cross-sectional area and moment of inertia are constant. Prior to investigating posed problem, another question arises: Which polynomial expressions to consider as candidates for the mode shape?

2. CANDIDATE MODE SHAPES

Since the governing differential equation for the inhomogeneous beams

$$\frac{d^2}{dx^2}\left(E(x)I\frac{d^2Y}{dx^2}\right) - \rho A\omega^2 Y = 0 \tag{13}$$

is of the fourth order, we ought to impose four boundary conditions (Note that for $E(x)$ =const, Eq. (13) reduces to Eq. 3). It makes sense therefore, to look for the polynomial of the fourth degree

$$Y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \tag{14}$$

We will perform a derivation for the $S - S$ case, while the other cases are dealt by in perfect analogy.

In this case the boundary conditions for $Y(x)$ read:

$$\begin{aligned} &Y = 0 \quad at \quad x = 0 \quad \text{and} \quad x = L \\ &EId^2Y/dx^2 = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L \end{aligned} \tag{15}$$

Satisfaction of the condition $Y(0) = 0$ results in the vanishing of the coefficient a_0. The condition $EIY''(0)=0$ leads to $a_2 = 0$. The remaining conditions yield

$$a_1L + a_3L^3 + a_4L^4 = 0 \tag{16}$$

$$6a_3L + 12a_4L^2 = 0 \tag{17}$$

From Eq. (17)

$$a_3 = -2a_4L \tag{18}$$

Substitution into Eq. (16) leads to

$$a_1 = a_4L^3 \tag{19}$$

The mode shape $Y(x)$ becomes

$$Y(x) = a_4\left(xL^3 - 2Lx^3 + x^4\right) \tag{20}$$

Thus we pose a somewhat puzzling inquiry of finding an inhomogeneous beam whose mode shape is given by Eq. (20).

3. SOLUTION OF THE INVERSE PROBLEM

We will confine ourselves with the simplest case when the inhomogeneity is restricted to the elastic modulus only; i.e.

$$E = E(x), \quad \rho = \text{const} \tag{21}$$

We immediately note that in order the polynomial solution to exist, it is necessary both terms in Eq. (13) are to be represented by polynomial expressions, after substituting into them Eq. (20). This implies we ought to look in the class of polynomial variations of $E(x)$. The question is how to determine its order, in order it to be compatible with the demanded fourth order polynomial expressions of the mode shape.

The second term in Eq. (13), i.e. $-\rho A\omega^2 Y$ is a fourth order polynomial since $Y(x)$ is such a quantity. Since the first term contains four derivatives in order it to constitute, fourth order polynomial $E(x)$ itself ought to be a fourth order polynomial:

$$E(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4. \tag{22}$$

We consider the case of the beam of constant mass density ρ=const. We substitute Eqs. (20) and Eq. (22) into Eq. (13), to get

$$\begin{aligned} &12I\left[2\left(b_0 - b_1L\right) + 6\left(b_1 - b_2L\right)x + 12\left(b_2 - b_3L\right)x^2 + 20\left(b_3 - b_4L\right)x^3 + 30b_4x^4\right] \\ &- \rho A\omega^2\left(xL^3 - 2Lx^3 + x^4\right) = 0 \end{aligned} \tag{23}$$

Which could be re-cast as a polynomial equation in increasing order of degrees of x:

$$\begin{aligned} &24I\left(b_0 - b_1L\right) + \left[72I\left(b_1 - b_2L - \rho A\omega^2L^3\right)x\right] + 144I\left(b_2 - b_3L\right)x^2 \\ &\quad + \left[240I\left(b_3 - b_4L\right) + 2\rho A\omega^2\right]x^3 + \left(360Ib_4 - \rho A\omega^2\right)x^4 = 0. \end{aligned} \tag{24}$$

Since this equation is valid for each value of x, the following algebraic equations must be satisfied:

$$24I\,(b_0 - b_1L) = 0, \tag{25}$$

$$72I\,(b_1 - b_2L) - \rho A\omega^2 L^3 = 0, \tag{26}$$

$$144I\,(b_2 - b_3L) = 0, \tag{27}$$

$$240I\,(b_3 - b_4L) + 2\rho AL\omega^2 = 0, \tag{28}$$

$$360Ib_4 - \rho A\omega^2 = 0. \tag{29}$$

We get five inhomogeneous equations with six unknowns, b_0, b_1, b_2, b_3, b_4, and ω^2. The system has infinite amount of solution. We set one of the coefficients, namely, b_4 as an arbitrary constant. The natural frequency squared ω^2 then becomes, from Eq. (29):

$$\omega^2 = 360b_4I/\rho A. \tag{30}$$

Substitution of Eq. (30) into Eq. (28) results in b_3 :

$$b_3 = -2b_L. \tag{31}$$

Equation (27) suggests that

$$b_2 = b_3L = -2b_4L^2. \tag{32}$$

Bearing in mind Eqs. (29) and (32) we get from Eq. (26):

$$b_1 = 3b_4L^3. \tag{33}$$

Finally, Eq. 25 yields

$$b_0 = b_1L = 3b_4L^4. \tag{34}$$

Thus, the coefficients b_0, b_1, b_2, and b_3, are expressed in terms of b_4 . The function describing the elastic modulus in Eq. (22) becomes:

$$E(x) = b_4\left(3L^4 + 3L^3x - 2L^2x^2 - 2Lx^3 + x^4\right) \tag{35}$$

This distribution of the modulus of elasticity can be rewritten in the different form of we introduce a coordinate z measured from the middle the beam's span:

$$x = z + L/2 \tag{36}$$

The elastic modulus becomes in terms of z

$$E(z) = \left(\frac{69}{16}L^4 - \frac{7}{2}L^2z + z^4\right)b_4 \tag{37}$$

Since only even powers of z are present, we conclude that

$$E(z) = E(-z) \tag{38}$$

In other words, the distribution of the elastic modulus is symmetric with respect to the middle cross-section. Also, the elastic modulus in Eq. (37) can be rewritten as

$$E(z) = \left[\left(z^2 - \frac{7}{4}L^2\right)^2 + \frac{5}{4}L^4\right]b_4 \tag{39}$$

which takes positive values for any z. This establishes that the derived solution confirms with the physical realizability of the above elastic modulus distribution due to its positiveness.

4. CONCLUSION

Simple solution presented in this paper appears to be ideal for the classroom setting. It takes only one hour to presentation, yet students can learn a lot about direct, inverse, and semi-inverse problems. This material enhances both the appreciation and understanding of the fundamentals of vibrations theory.

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MỘT VẤN ĐỀ SẼ PHÙ HỢP VỚI TẤT CẢ CÁC SÁCH GIÁO KHOA VỀ DAO ĐỘNG CƠ HỌC TRONG TƯƠNG LAI

Giới thiệu vấn đề cho các sách giáo khoa về dao động cơ học trong tương lai. Về cả phương diện toán học và phương diện nguyên lý nó đơn giản hơn hầu hết các vấn đề đã được nêu trong các sách giáo khoa hiện có. Điều này phục vụ cho bài toán ngược trong dao động của dầm không đồng nhất, có nghĩa dầm với module đàn hồi thay đổi dọc theo trục tọa độ. Đặc biệt trình bày lời giải của bài toán sau: Tìm phân bố của module đàn hồi của một dầm không đồng nhất sao cho dầm sẽ có dạng dao động là dạng đa thức đơn giản được chọn trước.